Arbitrages and Arrow-Debreu Prices

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The goal of this work is to check that there are no arbitrage opportunities in the CBOE market for S&P500 options and to extract from these options' quotes the state-price density consistent with the Merton model. The structure of the article is as follows: in Section 2 we examine the relations between arbitrages and Arrow-Debreu prices; in Section 3 we consider two models which seem to be consistent with the market prices of index options: the CEV model and the Merton model; finally, in Section 4 we estimate the state-price density consistent with the Merton-Geske model. Some conclusions follow. [JEL Classification: G13]

Key words: state-price density, index options, Merton-Geske model

1. - Introduction

One of the central ideas of economic thought is that prices contain useful information for economic decisions. According to Friedrich von Hayek (1945), the role played by the price network is to aggregate the single elements of information which is fully distributed, and synthesize them in a single statistic – the price. The price is what economic agents must know (besides their own specific information) in order to take correct decisions.

The most elementary prices are the Arrow-Debreu prices, *i.e.*

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the prices of state-securities (sometimes called state-contingent claims, pure securities or Arrow-Debreu securities), securities that pay off one unit of the *numéraire* if and only if a particular "state of the world" (or "state of nature") occurs at some point of time.[1]

In "nuclear" financial economics, the elementary particles are represented by Arrow-Debreu securities.[2] Even if they were created in a purely theoretical framework, these securities can be used to deal with practical applications:[3]

> *Given the enormous informational content that Arrow-Debreu prices possess and the great simplification they provide for pricing complex state-contingent securities such as options and other derivatives, it is unfortunate that pure Arrow-Debreu securities are not yet traded on any organized exchange.*

The wishes expressed in the above quote are now coming true with the ever growing diffusion of prediction markets (sometimes called information markets or event markets), *i.e.* markets where contracts on specific events are traded. The archetype of these markets is given by the Iowa Electronic Market (IEM), cited by Vernon Smith in his Nobel lecture.[4]

In the IEM, state-securities on electoral events and target rates of the Federal Reserve are traded. For instance, in the year 2008, participants were able to place a bet at price P_O that would pay \$1 if Barack Obama were elected and \$0 if not, or otherwise to place a bet at price P_M that would pay \$1 if John McCain were elected and \$0 if not.[5]

In our example, P_O and P_M can be referred to as "Arrow-Debreu (A-D) prices" or state-prices, *i.e.* prices of A-D securities, even if this is not strictly correct. Arrow-Debreu securities pay off \$1 only in a single state of the world and \$0 in all the remaining states, while the IEM does not trade securities which would pay

[1] See ARROW K.J. (1953).
[2] See SHARPE W. (1995).
[3] See AÏT-SAHALIA Y. - LO A.W. (1998).
[4] See SMITH V. (2002, pages 518-519).
[5] See RUBINSTEIN M. (2006, page 27).

off only if Obama won, the FED target rate was 1%, the S&P500 was 1,500, etc.

Arrow-Debreu securities are not generally traded, but they are often embedded in standard securities – from which they may sometimes be stripped. Therefore, the Arrow-Debreu prices are determined implicitly by traders to form the prices of securities traded in capital markets.

One of the most interesting fields of financial economics concerns the techniques used to extract information from the prices of securities and derivatives. The market is like a tool that continuously interrogates millions of people on their subjective probabilities and risk attitudes to synthesize the poll's results in the form of prices. By solving the so-called inverse problem, it is possible to extract the estimates of these probabilities and attitudes from market prices.[6]

In fact, our goal is to extract the Arrow-Debreu prices from the quotes of S&P500 options listed on the Chicago Board Options Exchange.

The structure of the article is as follows: in Section 2 we examine the relations between arbitrages and Arrow-Debreu prices. This is important because, according to the first fundamental theorem of financial economics, Arrow-Debreu prices exist if and only if there are no arbitrage opportunities.[7] Therefore, before estimating Arrow-Debreu prices, it is necessary to check that the data are arbitrage-free.

In Section 3 we consider two models which seem to be consistent with the market prices of index options: the constant-elasticity-of-variance (CEV) model and the Merton model.

Finally, in Section 4 we estimate the parameters of the Merton-Geske model. Some conclusions follow.

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[6] See RUBINSTEIN M. (1999, pages 13-19).
[7] See RUBINSTEIN M. (19, page 17). See also ROSS S.A. (2005, Chapter 2).

2. - Arbitrages

2.1 *Definition and Examples*

The definition of arbitrage given by Philip H. Dybvig and Stephen A. Ross is as follows:[8]

> ***arbitrage.*** *An arbitrage opportunity is an investment strategy that guarantees a positive payoff in some contingency with no possibility of a negative payoff and with no net investment. By assumption, it is possible to run the arbitrage possibility at arbitrary scale; in other words, an arbitrage opportunity represents a money pump. A simple example of arbitrage is the opportunity to borrow and lend costlessly at two different fixed rates of interest. Such a disparity between the two rates cannot persist: arbitrageurs will drive the rate together.*

Since they are a precious source of income for those who detect them, it is difficult to discover evidence of arbitrage opportunities which have actually been exploited. Since these opportunities sometimes repeat themselves, arbitrageurs avoid giving information. A rare exception is represented by a clinical paper by Myron S. Scholes and Mark A. Wolfson, where the authors describe an arbitrage opportunity they actually exploited:[9]

> *Our investment strategy was simple. We discovered that many companies offered stockholders the right to buy additional shares at a discount, typically of 5.263% (or 5/95) from extant market prices. To qualify to buy this*

[8] See DYBVIG P.H. - ROSS S.A. (1996).

The first author who used the term "arbitrage" with a meaning similar to the modern one seems to have been Mathieu de La Porte in his treatise "La Science des Négocians et Teneurs de Livres" (1704). See *Le Trésor de la Langue Française Informatisé* (web site *atilf.atilf.fr/*), item "arbitrage". In this item, three main kinds of arbitrages are identified: "currency arbitrage" (*arbitrage sur monnaies*), exploitable when misalignments of bilateral exchange rates occurs, "spot-forward arbitrage" (*arbitrage comptant contre terme*), to exploit violations of the forward-spot parity, and "repo arbitrage" (*arbitrage en reports*), which also exploits violations of the forward-spot parity.

[9] See SCHOLES M.S. - WOLFSON M.A. (1989).

discount stock, one had only to hold at least one share of company stock in certificate form and sign up to participate in the discount stock-purchase program. The next step was to mail in a check for stock periodically. The company then issued, free of commissions, discount shares that could be sold in the market within a few weeks. With an investment of $200,000 we realized a profit of $421,000 (consisting of $163,800 of net discount income (the sum of all gross discounts less transaction costs), $182,600 of return on investment due to a general increase in stock prices, and $74,600 of abnormal return on investment beyond the net discount income). This profit is net of brokerage fees, hedging losses, and other transaction costs. Ninety percent of our activity occurred over less than two years.
For example, a J.P. Morgan shareholder could buy up to $5,000 of J.P. Morgan stock each month at a 5.263% discount. If the shareholder could immediately sell this stock at no cost, a sure profit of $263.16 would result on each transaction. We would have preferred J.P. Morgan's sending us a check of $263.16 each month to our having to mail in the check, buy shares, and then sell them at a later date. In fact, if we could have avoided the transaction costs incurred in undertaking these tasks, we would have been satisfied to receive somewhat less. If investment is undertaken once a month at a discount of 5.263%, the compound annual return exceeds 85% of the monthly investment amount.

Another arbitrage has been exploited by Stephen A. Ross:[10]

I once was involved with a group that specialized in mortgage arbitrage, buying and selling the obscure and arcane pieces of mortgage paper stripped and created from government pass-through mortgages (pools of individual home mortgages). I recall one such piece – a special type of "IO" – which, after extensive analysis, we found would

[10] See Ross S.A. (2005, page 3, footnote 2).

offer a three-year certain return of 37 percent per year. That was the good news. The bad news was that such investments are not scalable, and, in this case, we could buy only $600,000 worth of it, which, given the high-priced talent we had employed, barely covered the cost of the analysis itself. The market found an equilibrium for these very good deals, where the cost of analyzing and accessing them, included the rents earned on the human capital employed, was just about offset by their apparent arbitrage returns.

2.2 *A Numerical Example*

Now we will extend an example used by Mark Rubinstein to give a numerical illustration of the various concepts that intertwine with arbitrages: present value, subjective probabilities and risk-neutral probabilities, utility functions and risk-aversion coefficients, Arrow-Debreu prices (or state-prices), stochastic discount factors (or the pricing kernel), volatility bounds.

The example concerns the valuation of a homeowner earthquake insurance policy.[11] The policy has a 1-year maturity and the premium is paid immediately.

The valuation procedure consists of the following steps:

1. to declare the subjective probabilities for the various possible states of nature (represented by different earthquake intensities);

2. to choose a utility function (generally a function belonging to the family of power functions) and to declare the degree of risk aversion;

3. to determine the risk-adjustment factors to be used to adjust the subjective probabilities;

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[11] See RUBINSTEIN M. (1999, pages 2-13). Our first extension with respect to Mark Rubinstein's original proof deals with the analytical steps for obtaining the risk-adjustment factors, based on the utility function of the individual who buys the insurance policy. The other two extensions regard the pricing kernel and the volatility bound.

4. to calculate the risk-neutral probabilities as products of the subjective probabilities and the risk-adjustment factors;

5. to calculate the Arrow-Debreu prices as products of the risk-neutral probabilities and the discount factor;

6. to derive the insurance policy's value as the weighted sum of the Arrow-Debreu prices, with weights equal to the insurance policy's payoffs.

In addition, the example allows us to check:

1. the equivalence of the above method and the approach based on the pricing kernel [a synonym for stochastic discount factors, defined by the ratio between the Arrow-Debreu prices and the subjective probabilities];

2. a lower bound for the kernel's volatility.

It should be stressed that the derivation of risk-adjustment factors, state prices, *etc.* from the homeowner's utility function is correct only if the homeowner's utility function is the utility of the representative investor – the "hypothetical agent who holds the market portfolio of all assets."[12]

Payoffs and Subjective Probabilities

In Table 1 five, mutually exclusive and exhaustive, "states of nature" have been reported. These states, measured on the Richter scale, offer a full description of the world's relevant aspects.

Table 1

EARTHQUAKE INSURANCE POLICY:
PAYOFFS AND SUBJECTIVE PROBABILITIES

Richter Scale	*Damage*	*Payoff* ($)	*Subjective Probability*
0 – 4.9	None	0	0.850
5.0 – 5.4	Slight	750	0.100
5.5 – 5.9	Medium	10,000	0.030
6.0 – 6.9	Severe	25,000	0.015
7.0 – 8.9	Large	50,000	0.005

[12] See Dybvig P.H. - Ross S.A. (2003, page 620).

The policy's payoffs, *i.e.* the amounts, z, payable as compensation for the damage caused by the earthquake, increase as a function of the latter's intensity.

The subjective probabilities, π, are personal degrees of belief associated by the homeowner with each state. Naturally, for π's to be probabilities, they must all be non-negative real numbers and sum to one.

One might think that the present value of the insurance policy would be its expected payoff discounted to the present by the one-year risk-free rate of return, r:

$$V = \frac{E(z)}{1+r} = \frac{\sum \pi z}{1+r}$$

where E is the "expected value" operator.

So if r = 0.05, the value of the insurance policy would be:

$$V = \frac{\$0 + \$75 + \$300 + \$375 + \$250}{1.05} = \frac{\$1{,}000}{1.05} = \$952.38.$$

However, this approach fails to consider risk aversion.

Utility Function

To take risk aversion into account, one has to calculate the risk-adjustment factors, f, needed to modify the subjective probabilities. The procedure is as follows:

1. to determine the marginal utilities of wealth, U_W, for each single state;

2. to calculate the risk-adjustment factors, f, as products of the marginal utilities and an arbitrary constant c ($f \equiv cU_W$), so that the sum of probabilities, π^*, defined by:

(1) $$\pi^* = \pi f$$

is equal to one.

By means of this calibration, the π^* may be considered as probabilities. In fact, they are all non-negative (since the marginal utilities are positive) and sum to one (because of the arbitrary constant).

To apply this procedure, let's suppose that the wealth of the homeowner is \$100,000 in the first state of nature, which is the "rich" state, where he does not suffer any damage caused by earthquakes (Table 2). In the other states of nature, his wealth diminishes proportionally to the damage caused by the earthquakes (supposedly equal to the policy's payoffs).

TABLE 2

RISK-ADJUSTMENT FACTORS
AND RISK-NEUTRAL PROBABILITIES

Richter's Scale	*Payoff* (z)	*Subjective Probability* (π)	*Wealth* (W)	*Marginal Utility* ($U_W = W^{\gamma-1}$)	*Risk-Adjustment Factor* ($f = c\ U_W$)	*Risk-Neutral Probability* ($\pi^* = f\ \pi$)
0 – 4.9	0	0.850	100,000	0.00316	0.9936	0.845
5.0 – 5.4	750	0.100	99,250	0.00317	0.9974	0.100
5.5 – 5.9	10.000	0.030	90,000	0.00333	1.0474	0.031
6.0 – 6.9	25.000	0.015	75,000	0.00365	1.1474	0.017
7.0 – 8.9	50.000	0.005	50,000	0.00447	1.4052	0.007

Note: by hypothesis, the elasticity, γ, of the utility function is equal to 0.5. The arbitrary constant, c, needed to define the risk-adjustment factors, f, is equal to 314.2194.

For each state of nature, we can calculate the marginal utility of wealth. We make the hypothesis that the utility function, *U(W)*, belongs to the family of power functions:

$$U(W) = \frac{W^{\gamma}}{\gamma}$$

Let's suppose that the elasticity, γ, is equal to 0.5. Then the marginal utility of wealth is $W^{\gamma-1} = W^{-0.5}$. As reported by Table 2, the marginal utility of \$1 increases as wealth diminishes.

As mentioned earlier, the risk-adjustment factors, f, needed to modify the subjective probabilities, π, were derived by multiplying the marginal utilities, U_W, by an arbitrary constant, c. In order to drive the sum of probabilities $\pi^* \equiv f\ \pi$ to 1, we put:

(2) $$\sum \pi^* = \sum \pi f = \sum \pi c U_W = 1.$$

Solving *(2)* with respect to c gives:

$$c = \left(\sum \pi U_W\right)^{-1}.$$

The actual value of the insurance policy can be determined by using the following formula:

(3) $$V = \frac{E^*(z)}{1+r} = \frac{\sum \pi^* z}{1+r}$$

where E^* is the operator "expected value" in a risk-neutral world.

By using the probabilities π^*, the value of the insurance policy is equal to:

$$V = \frac{\$0 + \$74.80 + \$314.22 + \$430.26 + \$351.31}{1.05}$$

$$= \frac{\$1{,}170.59}{1.05} = \$1{,}114.85$$

The actual value (\$1,114.85) of the policy is much higher than the value (\$952.38) we obtained earlier when we did not take into account the homeowner's risk aversion.

The effects of risk aversion were absorbed by the probabilities π^*. Therefore, the expected payoff was discounted using the risk-free interest rate (without making any further adjustment for risk aversion). This is the reason why the probabilities π^* are called risk-neutral probabilities. It should be stressed that the risk-neutral probabilities are generally different from the subjective probabilities.

Arrow-Debreu Prices

The Arrow-Debreu prices (or state-prices), q, are equal to the product of the risk-neutral probabilities, π^*, and the discount factor, $1/(1 + r)$:

(4) $$q = \frac{\pi^*}{1+r}$$

By using the state-prices, the valuation formula *(3)* for the insurance policy becomes:

(5) $$V = \sum q z$$

Given the *q*'s of Table 3,

TABLE 3

ARROW-DEBREU PRICES AND THE PRICING KERNEL

Subjective Probability (π)	*Risk-Adjustment Factor* (f)	*Risk-Neutral Probability* ($\pi^* = f\pi$)	*Arrow-Debreu Price* [$q = \pi^*/(1 + r)$]	*Pricing Kernel* [$\varphi = q/\pi = f/(1 + r)$]
0.850	0.9936	0.845	0.80438	0.9463
0.100	0.9974	0.100	0.09499	0.9499
0.030	1.0474	0.031	0.02993	0.9975
0.015	1.1474	0.017	0.01639	1.0927
0.005	1.4052	0.007	0.00669	1.3383

the value of the insurance policy is:

$$V = \$0 + \$71.24 + \$299.26 + \$409.77 + \$334.58 = \$1{,}114.85$$

By writing the formula in this way, it is natural to interpret the *q*'s as prices of Arrow-Debreu securities which pay \$1 only if a certain state of nature occurs and \$0 otherwise.

In particular, it should be noted that:

$$\sum q = \frac{1}{1+r}$$

and that, in our example:

$$\sum q = 0.80438 + 0.09499 + 0.02993 + 0.01639 + 0.00669$$
$$= 0.95238 = 1/1.05$$

This equality has a simple economic interpretation: the value of a portfolio that pays \$1 in each and every state is the value of receiving \$1 for certain, or $1/(1 + r)$. Therefore, although they are non-negative, the state-prices q are not probabilities because they do not sum to one (unless, of course, $r = 0$).

The Pricing Kernel

What we have seen so far can be summarized as follows:

1. using subjective probabilities, π, considers only personal beliefs;

2. using risk-neutral probabilities, π^*, considers the joint effects of both beliefs and risk aversion;

3. using state-prices, q, simultaneously takes into account beliefs, risk aversion, and time.

The present value can be calculated in many different ways. The most frequent formulation is now is based on the pricing kernel, φ.

The pricing kernel, synonymous with stochastic discount factors, is defined by the ratio between the Arrow-Debreu prices, q, and the subjective probabilities, π:

(6) $$\varphi = \frac{q}{\pi}$$

By substituting $q = \varphi\pi$ in *(5)*, the value of the insurance policy is:

$$V = \sum z\varphi\pi \equiv E(z\varphi)$$

and, given the values reported in Table 3,

$$V = \$0 + \$71.24 + \$299.26 + \$409.77 + \$334.58 = \$1{,}114.85$$

Besides, if π and q in *(6)* are substituted by *(1)* and *(4)*, the pricing kernel, φ, can also be represented as the ratio between the risk-adjustment factor, f, and the riskless return, $1 + r$:

$$\varphi = \frac{1}{1+r}$$

Volatility Bounds

Let r_i be the rate of return of the i-th Arrow-Debreu security (i = 1, 2, ..., 5) and $E(s_i)$ the risk premium or excess rate of return with respect to the risk-free interest rate. Let s be the vector of excess returns, $s_i \equiv r_i - r$:

$$s = \begin{bmatrix} s_1 = r_1 - r \\ s_2 = r_2 - r \\ \dots \\ s_5 = r_5 - r \end{bmatrix}$$

If there are no arbitrage opportunities, the value of a portfolio which pays s_i, *i.e.* the rate of return r_i in exchange for r, must be null. Therefore:

$$E(\varphi s) = 0 \tag{7}$$

where φ is the pricing kernel.

Since, by definition, the covariance between φ and s is:

$$\begin{aligned} \text{cov}(\varphi, s) &= E\{[\varphi - E(\varphi)][s - E(s)]\} \\ &= E[\varphi s - \varphi E(s) - E(\varphi)s + E(\varphi)E(s)] \\ &= E(\varphi s) - E(\varphi)E(s) - E(\varphi)E(s) + E(\varphi)E(s) \\ &= E(\varphi s) - E(\varphi)E(s) \end{aligned}$$

equation *(7)* can be written as:

$$E(\varphi)E(s) + \text{cov}(\varphi, s) = 0 \tag{8}$$

so that:

$$E(s) = -\frac{1}{E(\varphi)}\text{cov}(\varphi, s)$$

$$|E(s)| = \frac{1}{E(\varphi)}|\text{cov}(\varphi, s)|.$$

Besides, since – by definition – the absolute value of the correlation coefficient must be lower than or equal to 1, we have:

$$\left|\operatorname{cov}(\varphi, s)\right| \leq \sigma_\varphi \sigma_s \tag{9}$$

so that:

$$\left|E(s)\right| \leq \frac{1}{E(\varphi)} \sigma_\varphi \sigma_s . \tag{10}$$

Finally, using *(10)* we can derive the lower bound of Hansen and Jagannathan (1990) for the volatility, σ_φ, of the kernel:[13]

$$\sigma_\varphi \geq \frac{E(\varphi)\left|E(s)\right|}{\sigma_s} . \tag{11}$$

TABLE 4

PRICING KERNEL AND VOLATILITY BOUND

Payoff (z)	*Subjective Probability* (π)	*Pricing Kernel* (φ)	*Weighted Kernel* ($\pi\varphi$)
0	0.850	0.9463	0.80438
750	0.100	0.9499	0.09499
10,000	0.030	0.9975	0.02993
25,000	0.015	1.0927	0.01639
50,000	0.005	1.3383	0.00669
		Mean	0.95238
		Standard deviation	0.03366

Rate of Return ($r_i = z/V - 1$)	*Excess Rate of Return* ($s_i = r_i - r$)	*Current Value* s_i ($s \times \varphi \times \pi$)	*Weighted Value* s_i ($s \times \pi$)	*Covariance between* φ *and* s [Cov(φ, s)]
-1.0000	-1.0500	-0.84460	-0.89250	0.00461
-0.3273	-0.3773	-0.03584	-0.03773	0.00006
7.9698	7.9198	0.23701	0.23759	0.01093
21.4245	21.3745	0.35035	0.32062	0.04532
43.8490	43.7990	0.29308	0.21899	0.08481
	Mean	0.00000	-0.15302	0.14573
	Standard deviation		4.3880	

In order to check whether the lower bound *(11)* holds for the data of our numerical example, first of all we have to calculate the mean and the standard deviation of the pricing kernel (Table 4).

[13] See HANSEN L.P. - JAGANNATHAN R. (1991).

These two values are, respectively, equal to:

$$E(\varphi) = 0.95238 \quad \text{and} \quad \sigma_\varphi = 0.03366$$

Then, using the following formula, we calculate the rates of return, r_i, of each single element of the policy:

$$r_i = \frac{z}{V} - 1$$

where z is the payoff of the policy and V is its current value.

Next we calculate the excess rates of return $s_i \equiv r_i - r$ and the expected value $E(\varphi s)$ which, as requested by Equation *(7)*, is actually null.

Then we calculate the mean, $E(s)$, and the standard deviation, σ_s, of the excess returns, which are, respectively, equal to:[14]

$$E(s) = -0.15302 \quad \text{and} \quad \sigma_s = 4.3880$$

The covariance between the pricing kernel, φ, and the excess rates of return, s_i, is:

$$\text{cov}(\varphi, s) = 0.14573$$

Now we can check both *(8)* and *(9)*:

$$0.95238 \times (-0.15302) + 0.14573 = 0$$

$$|0.14573| \leq 0.03366 \times 4.3880 = 0.14770$$

Besides, since:

[14] Note that the risk premium, defined as the mean of the excess rates of return, $E(s)$, is negative. The reason why the homeowner buys the policy, even if it offers a negative risk premium, is that the risk of his hedged position is lower and his utility is higher.

$$0.03366 \geq \frac{0.95238 \times |-0.15302|}{4.3880} = 0.03321$$

the volatility bound *(11)* also holds.

3. - Arrow-Debreu Prices

3.1 *State-Price Density (SPD)*

The *state-price density* (SPD), *i.e.* the density function of Arrow-Debreu prices, can be estimated from the prices of marketable securities, as has been suggested by Ross (1976), Banz-Miller (1978) and Breeden-Litzenberger (1978).[15]

In the Black-Scholes-Merton model, where it is assumed that the dynamics followed by the stock price is given by an Itô process (a geometric Brownian motion), *i.e.* a stochastic process in continuous time for a continuous variable, the state-price density is continuous and the state-securities pay \$1 if the state is between x and $x + dx$.

Breeden-Litzenberger (1978) proved that the state-price density is equal to the second derivative of option prices with respect to the exercise prices.[16] It follows that the density function of state-prices can be estimated by observing the quotes of butterfly spreads, *i.e.* the portfolios made up of two long options with extreme exercise prices and two short options with the same intermediate exercise price.[17]

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[15] See ROSS S.A. (1976); BANZ R.W. (1978) and BREEDEN D.T. - LITZENBERGER R.H. (1978).

[16] These two authors give credit to Fisher Black for making the same discovery, but add that "the result was noted [by Black] as a mathematical curiosity rather than being derived as a general proposition." See BREEDEN D.T. - LITZENBERGER R.H. (1978, page 627, note 7).

[17] In fact, the second derivative of the call's price with respect to the exercise price is the limit of the incremental ratio $[c(K - \Delta K) - 2c(K) + c(K + \Delta K)]/(\Delta K)^2$. If $K_1 = K - \Delta K$, $K_2 = K$ and $K_3 = K + \Delta K$, with $\Delta K = 1$, the incremental ratio is equal to the value of the butterfly spread: $c(K_1) - 2c(K_2) + c(K_3)$.

3.2 *Volatility Smiles*

As is well known, the implied volatilities are the values of σ which, put in the Black-Scholes-Merton formula, make the theoretical values of the options equal to their market prices. If the Black-Scholes-Merton formula is valid, the implied volatilities should be constant, *i.e.* they should not change as a function either of the exercise price or of the options' maturity. This is not what actually happens in the options markets.

The actual patterns of volatility surfaces, *i.e.* the charts which show the implied volatilities as a function of options' maturities and exercise prices, are countless, as are the shapes of volatilities term structures and volatility smiles, *i.e.* the charts which show the implied volatilities as a function, respectively, of maturities and exercise prices.

The implied volatilities are such important variables for traders that options are sometimes quoted in terms of implied volatilities (under the hypothesis that the underlying model is the Black-Scholes-Merton model), instead of prices. Actually, traders prefer quoting implied volatilities to quoting option prices because implied volatilities are more stable and, as a result, their quotes do not change as frequently.

If there are no arbitrage opportunities, the volatility smile calculated with reference to the prices of calls must be equal to the volatility smile calculated with reference to the prices of puts. Otherwise, the put-call parity would not hold. Therefore, when traders refer to a certain relationship between implied volatilities and exercise prices, they do not have to specify which type of options they are referring to, since the same relationship must hold for calls and puts.

Generally, at least since 1987, the implied volatilities observed in the markets of index options have a shape similar to a skew or a smirk, *i.e.* a downward-sloping curve as a function of the exercise price.[18]

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[18] Instead, in the case of currency options, the implied volatility has a minimum for the at the money options and becomes higher and higher as the

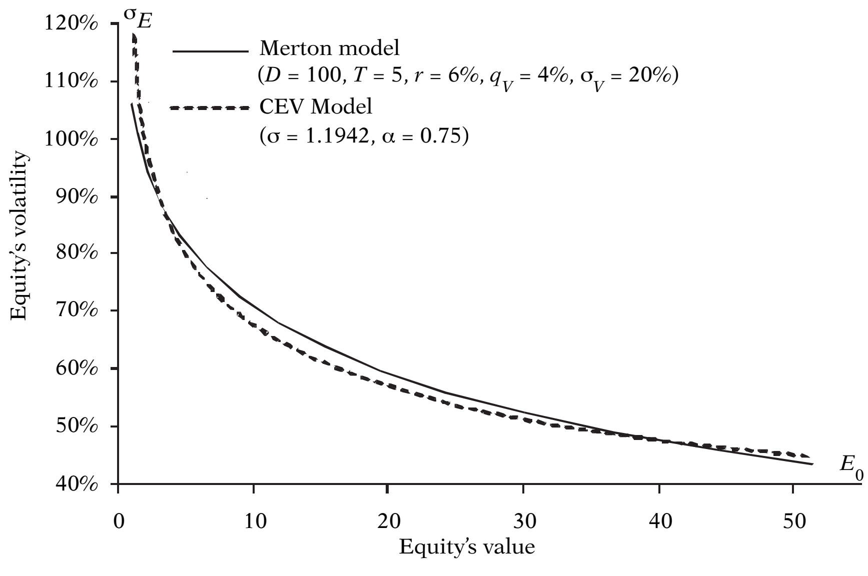
A model which can explain the downward-sloping shape of index options' volatility smiles is the constant-elasticity-of-variance (CEV) model, when the elasticity is negative. This model has been proposed by Cox (1975).[19]

Another model consistent with the negative slope of volatility smiles is the Merton (1974) model, where the dynamics of stock prices depends on the value of the firm's assets and the firm's leverage.[20]

If the stock volatilities of the Merton and CEV models are compared, the two functions appear reasonably similar, even if their convexity is different (Graph 1).

GRAPH 1

THE MERTON MODEL *VS* THE CEV MODEL: EQUITY'S VOLATILITY



options become in the money or out of the money. Therefore, the shape of the volatility smile is really similar to a "smile".

[19] See COX J.C. (1975).

[20] See MERTON R.C. (1974).

However, *ceteris paribus*, one may prefer the Merton model which, unlike the CEV model, permits an interesting interpretative key: if the stock price rises then the firm's debt-equity ratio decreases and this reduces the volatility of both earnings and stock prices.

3.3 *Compound Options*

In the Merton model, a common stock is a call option written on the firm's assets. As a consequence, call and put stock options are compound options (calls on a call or puts on a call). The stock options can therefore be valued by the Geske formulas (1977).[21]

If the stock is a call written on the firm's assets, with exercise price D and maturity T_D, then the value of a European call or put option, with exercise price K and maturity T, written on the stock is given by:

$$
\begin{aligned}
c &= V_0 e^{-q_V T_D} M(a_1, b_1; \sqrt{T/T_D}) - D e^{-rT_D} M(a_2, b_2; \sqrt{T/T_D}) \\
&\quad - e^{-rT} K\, N(a_2)
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
p &= D e^{-rT_D} M(-a_2, b_2; -\sqrt{T/T_D}) - V_0 e^{-q_V T_D} M(-a_1, b_1; -\sqrt{T/T_D}) \\
&\quad + e^{-rT} K\, N(-a_2)
\end{aligned}
$$

where

$$
a_1 = \frac{\ln(V_0 / V^*) + (r - q_V + \sigma_V^2 / 2)\, T}{\sigma_V \sqrt{T}}; \qquad a_2 = a_1 - \sigma_V \sqrt{T}
$$

$$
b_1 = \frac{\ln(V_0 / D) + (r - q_V + \sigma_V^2 / 2) T_D}{\sigma_V \sqrt{T_D}}; \qquad b_2 = b_1 - \sigma_V \sqrt{T_D}.
$$

M is the cumulative probability in a standardized bivariate

---

[21] See GESKE R. (1977).

normal distribution that the first variable is less than a and the second variable is less than b, when the coefficient of correlation between the variables is ρ;

V^* is the critical value of the firm's assets (at time T) which results in the stock price being equal to K. If the value of the firm's assets is higher than V^*, then the call should be exercised. Otherwise, it is the put which should be exercised.

Example 3.1

Let V_0 = 891.9441, D = 1,000, T_D = 5, q_V = 0.3%, σ_V = 2.3%, r = 5%; K = 100, T = 1, where T_D is the debt's maturity and T is the maturity of a European call, with exercise price K, written on a stock issued by the firm (Graph 2).

GRAPH 2

MERTON-GESKE MODEL: VALUE OF A EUROPEAN CALL

Stock Options (Merton-Geske Model)	
Asset value (V_o)	891.9441
Face value of debt (D)	1,000.0000
Time to maturity of debt (T_D)	5
Payout rate (q_V)	0.30%
Volatility of assets (σ_V)	2.30%
Risk-free rate (r)	5.00%
Strike price of stock option (K)	100
Time to maturity of stock option (T)	1
Stock value (E_0)	100.0000
Dividend yield (q_E)	2.68%
Volatility of stock (σ_E)	20.03%
Value (*cc*, *cp*, *pc*, or *pp*)	10.6481 call on a call

In this case, the stock value is 100, the dividend yield is 2.68 percent and the stock's volatility is 20.03 percent. The value of the stock option (a call on a call) is 10.6481.

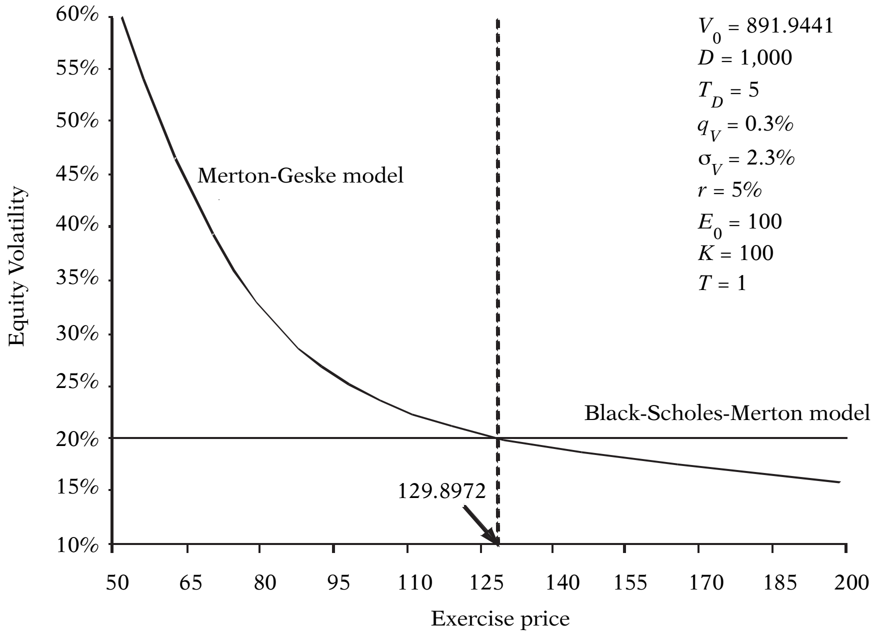
To calculate the volatility smile which one would observe in the market if traders were using the Merton-Geske model, we have first to determine the theoretical values of call options with different strikes, as in the following table:

Strike (K)	Merton-Geske Model		Black-Scholes-Merton Model	
	Theoretical value (c)	Implied volatility (σ)	Theoretical value (c)	Implied volatility (σ)
50	52.45286	63.08%	49.79878	20.03%
70	33.74898	40.31%	30.95273	20.03%
90	17.09809	27.97%	14.62012	20.03%
100	10.64812	24.78%	8.84756	20.03%
130	1.24236	20.02%	1.24458	20.03%
160	0.03974	17.69%	0.10879	20.03%
200	0.00005	15.73%	0.00291	20.03%

The volatilities implied by the theoretical values are reported in Graph 3. As can be seen, the volatility smile has the typical negative slope. On the contrary, the volatility smile would be flat at the 20.03 percent level if traders were using the Black-Scholes-Merton model. The implied volatilities of the two models cross at K = 129.8972.

GRAPH 3

MERTON-GESKE *VS* BLACK-SCHOLES-MERTON: VOLATILITY SMILE



3.4 *The Merton Model and Arrow-Debreu Prices*

In the Merton model, the state variable is V_T. The Arrow-Debreu prices are defined by the following state-price density:

$$\varphi(V_T) = \frac{e^{-rT} N'(d_2)}{V_T \sigma \sqrt{T}} \tag{13}$$

where:

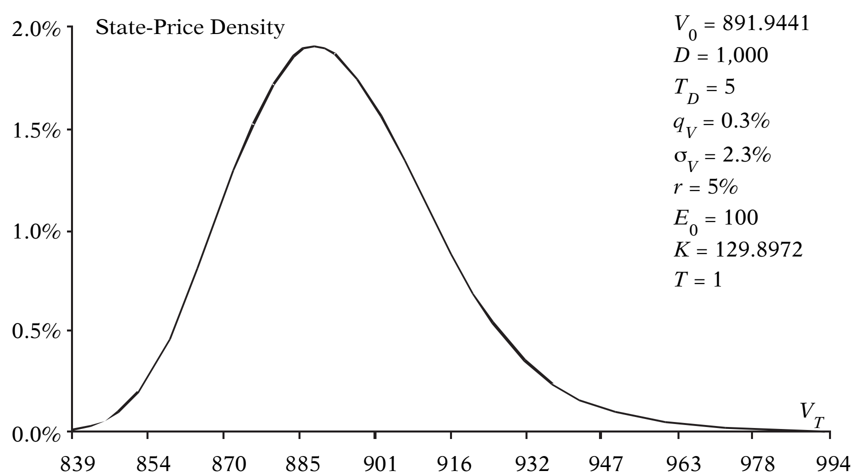
$$d_2 = \frac{\ln(V_0 / V_T) + (r - q_V - \sigma_V^2 / 2) T_D}{\sigma_V \sqrt{T_D}}$$

and N' is the density function of a standardized normal variable.

The state-price density *(13)* for Example 3.1 (when K = 129.8972) is shown in Graph 4.

GRAPH 4

MERTON MODEL: STATE-PRICE DENSITY



4. - Estimate of the Merton-Geske Model

As mentioned in the last section, the state-price density can be estimated from the prices of marketable securities. In this section we present the results of an empirical test serving to estimate the state-price densities, for various maturities, implied by the quotes of the S&P 500 index options traded on the Chicago Board Options Exchange, one of the most efficient markets. The state-price densities were estimated consistently with the Merton-Geske model.

4.1 *Arbitrage Opportunities*

Before estimating the state-price density consistent with the Merton-Geske model, we made an extensive empirical test, based on intraday data, to detect arbitrage opportunities. The algorithm we used, which solves a linear programming problem, was suggested by Mark Garman.[22]

Garman calls "hedge portfolios" the portfolios made by long and short positions on forwards, calls and puts, with the same maturity, written on the same asset.

Since a long forward is similar to a long call (or a short put) with a unit probability of exercise and a short forward is similar to a short call (or a long put) with a unit probability of exercise, all the elements of the portfolio are called "options", without any distinction. The options considered are all European.

The payoff of a hedge portfolio is a piecewise linear function of the final value of the underlying asset, given the typical shape of the payoffs of calls, puts, and forwards.

In general, any piecewise linear function with n breakpoints [K_0 (= 0), K_1, ..., K_{n-1}] can be disentangled using only two basic functions, the Heaviside "step function" [so called because of the

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[22] See GARMAN M.B. (1976). A revision of Garman's algorithm, aimed to improve its efficiency, has been proposed in my B.S. thesis. See BARONE G. (2005, pages 27-48).

English mathematician Oliver Heaviside (1850-1925)] and the "ramp function".

The step function makes it possible to measure the jump of the piecewise linear function at the breakpoint, while the ramp function quantifies the slope's change at that point.

Garman uses six vectors to describe profits (or losses) on the six basic positions (long call, short call, long put, short put, long forward, short forward).

Each vector contains four parameters: the first two multiply the values of the step function at breakpoints $K_0 = 0$ and $K_1 = K$, while the last two multiply the values of the ramp function at the same breakpoints.

Garman also defines some surplus variables, whose values are strictly positive only when the hedge portfolio gives a profit. Finally, he sets up a linear programming problem where one has to maximize the sum of surplus variables under the constraint that the portfolio never generates losses.

In the following two examples we show two arbitrage opportunities detected by our Garman-based software.

Example 4.1

The first arbitrage opportunity could have been exploited on April 11, 2007 (11:54 ECT). The following quotes for futures and options maturing on June 20, 2008, were observed:

Contract	*Strike*	*Bid*	*Ask*
Futures	–	1,493.8	1,495.8
Call	1,000	471.4	474.0
Put	1,000	3.9	4.9
Call	1,600	35.0	37.4
Put	1,600	130.3	132.9

The Garman algorithm detected the following arbitrage portfolio:

Contract	*Position*	*Quantity*	*Strike*	*Price*	*Inflows (Outflows)*
Futures	long	1.000000	–	1,495.8	0.000000
Call	short	-0.173667	1,000	471.4	81.866467
Put	long	0.173667	1,000	4.9	-0.850967
Call	short	-0.826333	1,600	35.0	28.921667
Put	long	0.826333	1,600	132.9	-109.819700

The arbitrage profit is equal to the algebraic sum of inflows and outflows associated with the five contracts in the portfolio

$$\begin{aligned}&\$0+\$81.866467-\$0.850967+\$28.921667\\&-\$109.8197=\$0.1174667\end{aligned}$$

Obviously, by magnifying the "scale" of the portfolio – for given prices – the profit would have been a multiple of $0.1174667.

The arbitrage portfolio is riskless: its final value is always null, whatever the level of the S&P 500 at the contracts' maturity. This is shown in the following table.

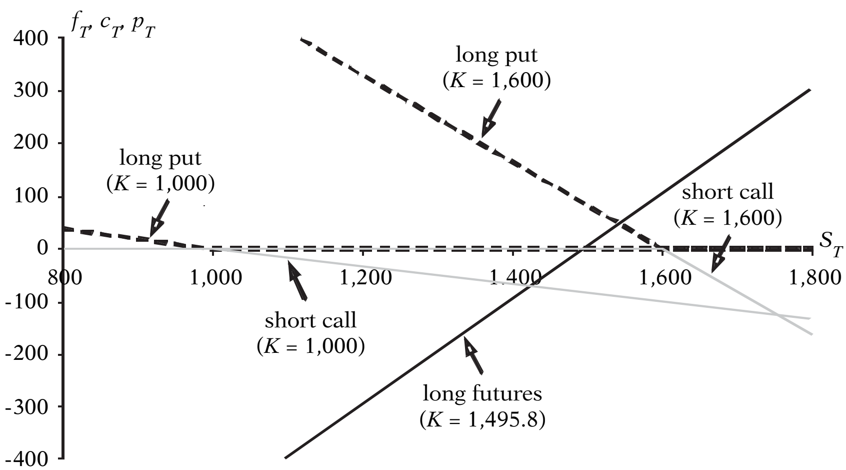
Contract	*Quantity*	$S_T < 1{,}000$	$1{,}000 \le S_T < 1{,}600$	$1{,}600 \le S_T$
f_0	1.000000	S_T - 1,495.8	S_T - 1,495.8	S_T - 1,495.8
$-c_1$	-0.173667	0	-(S_T-1,000)×0.173667	-(S_T-1,000)×0.173667
p_1	0.173667	(1,000-S_T)×0.173667	0	0
$-c_2$	-0.826333	0	0	-(S_T-1,600)×0.826333
p_2	0.826333	(1,600-S_T)×0.826333	(1,600-S_T)×0.826333	0
Total		0	0	0

The absence of risk can also be shown graphically.

In Graph 5, there are five thin lines which show, with some overlapping, the final value of the contracts included in the arbitrage portfolio and one thick line, lying on the horizontal axis, which shows the final value of the whole portfolio. The arbitrage portfolio is made up of a long futures and a short synthetic forward, which have the same maturity and the same delivery price. The short synthetic forward, whose delivery price is equal to 1,495.8 (= 0.173667 × 1,000 + 0.826333 × 1,600), was constructed as the weighted average of two synthetic short forwards, whose delivery prices are equal to 1,000 and 1,600, respectively.

Graph 5

FINAL VALUE OF AN ARBITRAGE PORTFOLIO (*Example 4.1*)



Example 4.2

Another arbitrage opportunity could have been exploited on the same day, at 10:24, when the following quotes for contracts maturing on December 19, 2008 were observed:

Contract	*Face Value /Strike*	*Bid*	*Ask*
Bond	1,000	917.4939	919.0645
Futures	–	1,522.8	1,524.8
Call	600	843.6	846.2
Put	600	0.2	0.6

The Garman algorithm detected the following arbitrage portfolio:

Contract	*Position*	*Quantity*	*Face Value/Strike*	*Price*	*Inflows (Outflows)*
Bond	short	-0.922800	1.000	917.4939	846.663369
Futures	short	-1.000000	–	1,522.8	0
Call	long	1.000000	600	846.2	-846.2
Put	short	-1.000000	600	0.2	0.2

The arbitrage profit is equal to the algebraic sum of inflows and outflows associated with the four contracts in the portfolio

$$\$846.6634 + \$0 - \$846.2 + \$0.2 = \$0.663369$$

Obviously, by magnifying the "scale" of the portfolio – for given prices – the profit would have been a multiple of \$0.663369.

The arbitrage portfolio is riskless: its final value is always null, whatever the level of the S&P 500 at the contracts' maturity. This is shown in the following table:

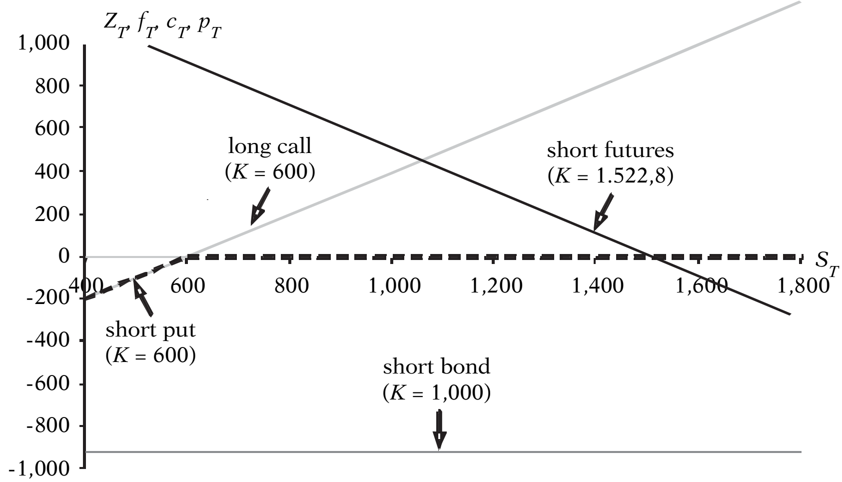
Contract	*Quantity*	$S_T < 600$	$600 \leq S_T < 1{,}000$	$1{,}000 \leq S_T$
$-Z_0$	-0.922800	-0.922800 × 1,000	-0.922800 × 1,000	-0.922800 × 1,000
$-f_0$	-1.000000	$-(S_T \times 1{,}522.8)$	$-(S_T \times 1{,}522.8)$	$-(S_T \times 1{,}522.8)$
c	1.000000	0	$S_T \times 600$	$S_T \times 600$
$-p$	-1.000000	$-(600 - S_T)$	0	0
Total		0	0	0

The final value of the arbitrage portfolio and its components is shown in Graph 6.

The portfolio is made up of a short zero-coupon bond, whose final value is 922.8, and a long synthetic zero-coupon bond with the same value (922.8 = 1,522.8 – 600) obtained by going short on a futures with delivery price of 1,522.8 and long on a synthetic forward (long call plus short put) with delivery price of 600.

Graph 6

FINAL VALUE OF AN ARBITRAGE PORTFOLIO (*Example 4.2*)



4.2 *Estimation Procedure*

In order to estimate the parameters of the Merton-Geske model, we used the quotes of the S&P500 options listed on the Chicago Board Options Exchange on April 11, 2007 (12:04 ECT). At that date there were 442 listed calls and an equal number of puts. The implied volatilities showed clear smiles. For instance, the implied volatilities of the deep-in-the money calls maturing on June 15, 2007, were equal to about 20 percent while the implied volatilities of the corresponding deep-out-of-the money calls were equal to 7-8 percent (Graph 7).

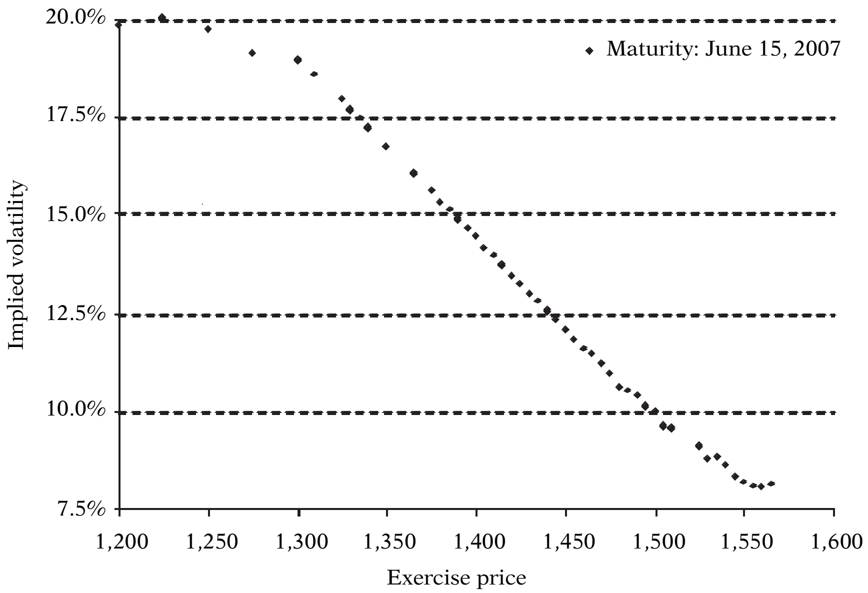
After checking that there were no arbitrage opportunities, and then *a fortiori* that the put-call parity held, we restricted the database and used only the calls to estimate the model's parameters.

The S&P500 options listed on the CBOE are European options written on the product of 100 and the index level. On April 11, 2007 (12:04 ECT), the index level was 1,442.9.

There were 14 option maturities: the shortest options matured

GRAPH 7

CBOE: VOLATILITY SMILE (APRIL 11, 2007)



after 2 days and the longest after 2 years and 8 months. There were 103 exercise prices, ranging from 600 to 2,000.

In order to estimate the model's parameters we used the Excel' Solver to minimize the standard least squares objective function: the sum of the squared deviations between the theoretical and actual values of the options.[23] Among the options we included the index, considered as a standard call, with exercise price D and maturity T_D, written on the assets of the S&P500 firms' basket.

The theoretical value, c, of the calls is given by the Geske formula *(12)*. The theoretical value, E_0, of the index is given by the standard formula of Black-Scholes-Merton. The risk-free interest rates for the maturities of options and debt have been determined in order to have a perfect fit with the prices of 3-

[23] "The routine works well provided that the spreadsheet is structured so that the parameters being searched for have roughly equal values." See HULL J.C. (2009, page 486).

month Eurodollar futures simultaneously observed at the Chicago Mercantile Exchange.[24]

The parameters to be estimated are:

V_0: the current value of assets;

D: the debt's face value;

T_D: the debt's maturity;

σ_V: the asset volatility;

q_V: the payout ratio of assets.

By-products of the estimation procedure are the theoretical level (E_0) of the S&P500 index, the index dividend yield (q_E) and the index volatility (σ_E).

First Trial

In our first trial, the problem of minimizing the objective function was solved by imposing three constraints:

1. the theoretical level of the S&P500 index to be equal to its actual level (1,442.9);

2. the index dividend yield to be equal to 1.88 percent (the estimate reported on the site *www.indexarb.com*);

3. the index volatility to be equal to 15.76 percent (the volatility implied by the longest at-the-money calls).

We did not find a solution.

Second Trial

In our second trial we imposed the constraint that the debt's face value, D, be equal to 1,000, a level comparable with the level of equity (1,442.9). We also set the debt's maturity, T_D, to 5 years, to be consistent with the standard used in the credit derivatives market.

The choice of a 5-year maturity for the debt is supported by the results obtained by Geske and Zhou (2009) in an extensive

[24] The continuously compounded rates for the maturities of the options smoothly decrease from the one-day 5.91% to the 32-month 4.95%. The appropriate rate for the 5-year maturity of the debt is 5.00%. The methodology for perfectly fitting the 3-month Eurodollar futures prices is shown in my M.S. thesis: see Barone G. (2007, pages 112-118).

empirical analysis of the Merton-Geske model.[25] These authors meticulously used each firm's balance sheet to find exogenous values for the debt and its duration.[26]

Instead, our estimates of V_0 = 2,351.12, q_V = 1.13 percent, σ_V = 9.67 percent are based on *ad hoc* values for both the debt and its duration (F = 1.000, T_D = 5). This is a key point which affects our results.

The debt's current value, equal to the difference between the asset value, V_0, and the equity value, E_0, was equal to 908.2 (= 2,351.12 – 1,442.9). Therefore, the share of the assets pertaining to bondholders was equal to the 38.6 (= 908.2 / 2,351.12) percent.

The estimate of q_V shows that 1.13 percent of the assets is used each year to pay the dividends to the shareholders.

The asset volatility, σ_V, is a measure of the business risk faced by the corporations. The level of σ_V, equal to 9.67 percent, is about 2/3 of the risk of a well-diversified financial investment, measured by the index volatility (15.76 percent).

The standard error, defined as the standard deviation of the differentials between the actual and theoretical values of the

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[25] The cited paper was unknown to me when I wrote the thesis which this essay is taken from. According to GESKE R. - ZHOU Y. (2009, page 9, text and footnote 12), the average duration of debt is close to 5 years: «The resultant average daily duration of aggregate debt in the aggregate balance sheet for the 500 firms in the S&P index is 4.71 years during the 100 month period in 1996-2004. ... We also find that the variation in aggregate debt duration is small and bounded between 4.5 and 5.1 years, respectively. This implies that our stylized model placing aggregate debt at a specified duration should work well».

[26] The nominal value of the firm's complex debt structure has been approximated by aggregating different balance-sheet items, classified according to their maturity. See GESKE R. - ZHOU Y. (2009, page 17): «The balance sheet information we collect comes from S&P's annual and quarterly Compustat. This book value of debt data is categorized as due in years 1 through 5 (Data 44, 91, 92, 93, 94), and greater than 5 (Data 9 minus items 91-94), which we place at 7 years. To these categories we add current liabilities (Data 5), deferred charges (Data 152), accrued expenses (Data 153), short term notes payable, deferred federal, foreign, and state taxes (Data 206, 269, 270, 271), all payable in year 1. All long-term debt tied to prime (Data 148) and debentures (Data 82), we place in year 7, respectively.[23] {This follows from Guedes and Opler (JF, v51, 5, 1996), page 1818, who provide evidence that the mean (of 7,362 issues) duration of long term US corporate debt is 7 years during the time period 1982-1993.} The debt due on each day in each quarter of each year for the S&P 500 firms is the sum of the debt due for all 500 firms for that day in that quarter of that year».

contracts, was equal to 17.7, *i.e.* 10.5 percent of the average price of the options (169.4).

Third Trial

To reduce the standard error, in our third trial we eliminated the three constraints for E_0, q_V and σ_V that we mentioned at page 24, while maintaining D = 1,000 and T_D = 5. After this, the standard error fell to 10.5 and the "unconstrained" estimates of the parameters were as follows: V_0 = 2,350.3 (compared with 2,351.12), q_V = 1.19 percent (1.13), σ_V = 7.01 percent (9.67), E_0 = 1,435.9 (1,442.9), q_E = 1.99 percent (1.88) and σ_E = 11.48 percent (15.76).

By decreasing the value and volatility of the assets and increasing the payout ratio, the algorithm reduced the theoretical value of the options (thus lowering the standard error) and brought the average error almost to zero, after it had been negative in the previous trials (thus signaling that the market was undervaluing the calls with respect to the model).

However, the errors continued to show a clear pattern, being positive for the shortest maturities and negative for the longest. According to the model, the quotes of the shortest-maturity calls should be lower and those of the longest calls should be higher.

Based on the put-call parity, the market underpricing of the longest calls entails the overpricing of the corresponding puts. It therefore seems that the market assigned a higher probability than the model to the reduction of stock prices for the longest maturities, consistently with the *crash-o-phobia* hypothesis of Rubinstein (1994):

> *... the Black-Scholes model worked quite well during 1986. ... However, during 1987 this situation began to deteriorate with percentage errors approximately doubling. 1988 represents a kind of discontinuity in the rate of deterioration, and each subsequent year shows increased percentage errors over the previous year. One is tempted to hypothesize that the stock market crash of October 1987 changed the way market participants viewed index options. Out-of-the-money*

> *puts … became valued much more highly, eventually leading to the 1990 to 1992 (as well as current) situation where low striking price options had significantly higher implied volatilities than high striking price options. ... The market's pricing of index options since the crash seems to indicate an increasing "crash-o-phobia,"* ... (pages 774-775)

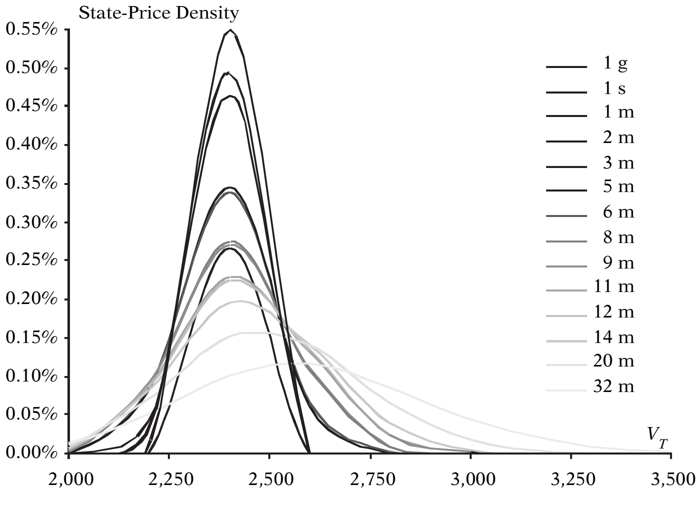
However, we should also consider the possibility that the quotes for long-maturity options do not signal information, since their liquidity is quite low. The arbitrage opportunities we detected support this hypothesis.

4.3 *Arrow-Debreu Prices*

Finally, by using the formula *(13)*, we estimated the state-price densities for the 14 maturities of the S&P500 options listed on the CBOE (Graph 8). Naturally, the flattest density functions refer to the longest maturities.

Graph 8

MERTON-GESKE MODEL (UNCONSTRAINED ESTIMATES): STATE-PRICE DENSITIES



5. - Conclusions

In this work we proposed a two-stage parametric method to extract information from the quotes of S&P500 options.

In the first stage, the linear-programming algorithm proposed by Mark Garman has been used to check for the absence of arbitrage opportunities. In the second stage, we estimated the parameters of the Merton model by using the Geske formula and considering the S&P500 index options as compound options (calls on a call or puts on a call), written on the value of the assets of the firms included in the S&P500 underlying basket. The estimates made it possible to calculate the state-price densities for all the options' maturities.

The Merton-Geske model, which is consistent with the hypothesis of an inverse relation between the level of equity prices and their conditional volatility, allows interesting indications to be extracted from options' quotes on the leverage and business risk of the underlying corporations. Our estimates represent a novelty for the literature on options, analogously to those reported by Geske and Zhou (2009) in an extensive empirical analysis which highlights the key role played by leverage in option pricing.

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